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THE GROWTH OR COLLAPSE OF A SPHERICAL BUBBLE IN A VISCOUS COMPRESSIBLE LIQUID

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TABLE OF CONTENTS

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ABSTRACT

With the help of a hypothesis first proposed by Kirkwood and Bethe, the partial differential equations for the flow of a compressible liquid surrounding a spherical bubble are reduced to a single total differential equation for the bubble-wall velocity. The Kirkwood-Bethe hypothesis represents an extrapolation of acoustic theory and would be expected to be most accurate when all liquid velocities are small compared to the velocity of sound in the liquid. However, the present theory is found to agree quite well with the only available numerical solution of the partial differential equations which extends up to a bubble-wall velocity of 2.2 times the sonic velocity.

In the particular case of a bubble with constant (or zero) internal pressure, an analytic solution is obtained for the bubble-wall velocity which is valid over the entire velocity range for which the Kirkwood-Bethe hypothesis holds. In the more general situation, when the internal pressure is not constant, simple solutions are obtained only when the bubble-wall velocity is considerably less than sonic velocity. These approximate integral solutions are obtained by neglecting various powers of the ratio of wall velocity to sonic velocity. The zero-order approximation coincides with the equations for a bubble in an incompressible liquid derived by Rayleigh; the first-order approximation agrees with the solution obtained by Herring using a different method. The second-order approximation is presented here for the first time.

The complete effects of surface tension, and the principal effects of viscosity, as long as the density variation in the liquid is not great, can be included in the analysis by suitably modifying the boundary conditions at the bubble wall. These effects are equivalent to a change in the internal bubble pressure. With this change, the same equations for the bubble-wall velocity are applicable to a viscous liquid with surface tension. Conditions under which the effects of surface tension and viscosity can be neglected are also determined.

First and second-order approximations to the velocity and pressure fields throughout the liquid are derived. From these expressions, the acoustic energy radiated is calculated.

I. INTRODUCTION

In connection with studies on cavitation and cavitation damage it is desirable to have mathematical expressions for the pressure and velocity fields in the neighborhood of a growing or collapsing gas or vapor-filled bubble in a liquid. Rayleigh[1, 2*] in 1917 solved the problem for a spherical bubble in an incompressible nonviscous liquid. In many practical bubble-collapse situations, however, it appears that local velocities reach an appreciable fraction of the velocity of sound in the liquid, and the compressibility of the liquid cannot safely be neglected. Herring[3, 4] in 1941 derived a better approximation, accurate to the first power of the ratio of liquid velocity to sonic velocity. His analysis, however, gives expressions for velocities and pressures only at the bubble wall and not throughout the liquid. Trilling[5] has recently obtained Herring's result by a somewhat simpler method, and he has also derived complicated integrals for obtaining the complete velocity and pressure fields with first-order accuracy. In the present paper, the analysis is generalized to include higher order compressibility terms, and also the effects of viscosity and surface tension. Moreover, it is found possible to simplify the resulting expressions to forms which permit convenient numerical calculation.

II. BASIC FLOW RELATIONS

If a spherical bubble grows or collapses in an infinite volume of liquid, and gravity and other asymmetric perturbing effects are neglected, the liquid flow will be spherically symmetric and hence irrotational. In any irrotational flow, the vector velocity, $\bar{u}$, can be written in terms of a velocity potential, ϕ:

$$\bar{u} = -\nabla\phi \tag{1}$$

while the equation for conservation of momentum[6] becomes

$$\frac{\partial}{\partial t}(-\nabla\phi) + (\bar{u}\cdot\nabla)\bar{u} = -\frac{\nabla p}{\rho} + \frac{4\mu}{3\rho}\nabla(\nabla\cdot\bar{u}) \, , \tag{2}$$

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*See bibliography at end of this report.

where p is the pressure, ρ the density and μ the viscosity of the liquid. In Eq. (2) it is assumed that the flow is irrotational and that μ is constant throughout space (but may vary with time).

The last term in Eq. (2) may be transformed by the use of the conservation of mass equation:

$$\nabla \cdot \bar{u} = - \frac{1}{\rho} \frac{D\rho}{Dt} , \tag{3}$$

where D/Dt is the "particle derivative" following the motion of the fluid. It is evident from Eq. (3) that the viscous term in Eq. (2) vanishes if either the viscosity or the compressibility of the liquid vanishes. In the present work, interest is centered on situations where the effect of viscosity is small,and of compressibility moderately small. It is reasonable, therefore, to neglect a term which represents the interaction between the two small effects*. Use of this approximation does not imply that all viscous effects are neglected; viscous dissipation occurs even in the incompressible situation, but the viscous terms appear only in the boundary condition, as will be shown below.

With this simplification, Eq. (2) can be integrated to give

$$- \frac{\partial \phi}{\partial t} + \frac{u^2}{2} = - \int_{p_\infty}^{p} \frac{dp}{\rho} , \tag{4}$$

provided that two assumptions are made. The first assumption is that the pressure p_∞, an infinite distance from the bubble, is constant and the velocity and velocity potential vanish at infinity, so that no constant of integration need appear in Eq. (4). It is apparent that any deviation from these conditions at infinity would either violate the condition of spherical symmetry or, if symmetric, propagate inward and attain an infinite amplitude at any finite distance from the bubble, due to spherical convergence. The second assumption is that the liquid density, ρ, can be expressed as a function of the pressure only. For isentropic flow, with no heat flow or viscous dissipation, this condition is exactly satisfied. Even if moderately strong thermal

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*In most real liquids, the variation of viscosity with pressure is relatively greater than the variation of density with pressure, and the term neglected here can be shown to be smaller than other terms due to variable viscosity which have already been omitted in Eq. (2).

and viscous processes are present, their effect on the density is usually negligible for liquids because the coefficient of thermal expansion for liquids is small.

It is convenient to denote the pressure integral by the symbol h:

$$h(p) = \int_{p_\infty}^{p} \frac{dp}{\rho} . \tag{5}$$

Thermodynamically, the quantity h is the enthalpy difference between the liquid at pressure p and at pressure p_∞.

Under the proper initial and boundary conditions, which will be discussed later, the flow field in the liquid will consist entirely of "outgoing" velocity and pressure waves. If all velocities were small compared to sonic velocity, and the sonic velocity did not vary significantly from its constant value, c_∞, at infinity, the well known expression for diverging spherical sound waves would be applicable:

$$\phi = \frac{1}{r} f(t - r/c_\infty) , \tag{6}$$

where r is the distance from the center of the bubble and f is an unspecified function of the argument $(t - r/c_\infty)$. Equation (4) could then be written

$$r (h + u^2/2) = f'(t - r/c_\infty) . \tag{7}$$

Equations (6) and (7) show that in the quasi-acoustic approximation* both the quantities $r\phi$ and $r (h + u^2/2)$ are propagated outward with a propagation velocity c_∞. As a more accurate approximation when liquid velocities attain appreciable fractions of the sonic velocity, it is plausible to assume either that $r\phi$ or that $r (h + u^2/2)$ is propagated outward with a variable velocity $(c + u)$, where c is the local

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*The adjective "quasi-acoustic" is used here because in conventional acoustics the further approximation that $u^2/2$ is negligible compared to h is made, and thus r h is found to propagate with velocity c_∞.

sonic velocity. The alternative assumptions, however, are only approximately equivalent. Kirkwood and Bethe, in their theory of underwater explosions[7], make the second of these assumptions, i.e., they assume that $r\,(h + u^2/2)$ is propagated with a velocity $(c + u)$. If this assumption is also used in the present analysis, then

$$\frac{\partial}{\partial t}\left[r\left(h + \frac{u^2}{2}\right)\right] = -\,(c + u)\,\frac{\partial}{\partial r}\left[r\left(h + \frac{u^2}{2}\right)\right]. \tag{8}$$

Equation (8) is more conveniently written in terms of the particle derivative, $D/Dt = \partial/\partial t + u\,\partial/\partial r$:

$$\frac{D}{Dt}\left[r\left(h + \frac{u^2}{2}\right)\right] + c\,\frac{\partial}{\partial r}\left[r\left(h + \frac{u^2}{2}\right)\right] = 0\,, \tag{9}$$

which can be expanded to give

$$r\,\frac{Dh}{Dt} + r\,u\,\frac{Du}{Dt} + (c + u)\left(h + \frac{u^2}{2}\right) + r\,c\,\frac{\partial h}{\partial r} + r\,c\,u\,\frac{\partial u}{\partial r} = 0\,. \tag{10}$$

In the spherically symmetric situation, the momentum relation, Eq. (2), becomes

$$\frac{Du}{Dt} = -\,\frac{\partial h}{\partial r}\,, \tag{11}$$

where the viscosity-compressibility interaction term is again neglected; while the continuity relation, Eq. (3), can be written

$$\frac{\partial u}{\partial r} + \frac{2u}{r} = -\,\frac{1}{c^2}\,\frac{Dh}{Dt}\,, \tag{12}$$

since $c^2 = dp/d\rho$ and $dp/\rho = dh$. If derivatives with respect to t are eliminated from Eq. (10) by means of Eqs. (11) and (12), one obtains the relation

$$r\,\frac{\partial h}{\partial r}\left(1 - \frac{u}{c}\right) + h\left(1 + \frac{u}{c}\right) - r\,c\,\frac{\partial u}{\partial r}\left(1 - \frac{u}{c}\right) - 2\,c\,u\left(1 - \frac{u}{4c} - \frac{u^2}{4c^2}\right) = 0\,. \tag{13}$$

This relation between the velocity and enthalpy fields throughout the liquid (or between the velocity and pressure fields, since enthalpy is a function of pressure) must hold at any instant, if the assumptions of the theory are valid. In particular, the initial conditions specified for a given spherical flow situation must satisfy Eq. (13). If initial velocity and pressure fields are specified which do not satisfy Eq. (13), the resulting flow will include converging as well as diverging spherical waves, and the present theory will not be applicable. It may be noted that the simple initial condition that the liquid is undisturbed, $p = p_\infty$, $h = 0$, $u = 0$, satisfies Eq. (13).

Equation (13) can be rearranged to give, after division by $r\,c(1 - u/c)$.

$$\frac{\partial u}{\partial r} + \frac{2u}{r}\left(\frac{1 - u/4c + u^2/4c^2}{1 - u/c}\right) = \frac{1}{c}\frac{\partial h}{\partial r} + \frac{1}{c}\frac{h}{r}\left(\frac{1 + u/c}{1 - u/c}\right). \quad (14)$$

In situations where the effective wavelength of the motion is much less than the radial co-ordinate of the region of interest, then $|u/r| << |\partial u/\partial r|$, $|h/r| << |\partial h/\partial r|$ and Eq. (14) becomes approximately

$$\frac{\partial u}{\partial r} = \frac{1}{c}\frac{\partial h}{\partial r}\,, \quad (15)$$

except in the neighborhood of $u = +c$, where the factors in parentheses become large. For such short wavelengths, the relation between velocity and pressure in the wave should be practically the same as for a plane wave. In progressive plane waves of finite amplitude (arbitrarily large velocities and pressures), Riemann's method[2] shows that

$$u = \int_{p_\infty}^{p} \frac{dp}{\rho c} = \int_{o}^{h} \frac{dh}{c}\,. \quad (16)$$

Differentiation of Eq. (16) gives Eq. (15). It follows that the Kirkwood-Bethe hypothesis is accurate in the limit of small wavelengths and arbitrary velocities (except when $u \approx +c$), as well as for the small velocities and arbitrary wavelengths for which it was derived.

A particle-derivative relation for spherical flow is obtained by using Eqs. (11) and (12) to eliminate derivatives with respect to r from Eq. (10):

$$r \frac{Dh}{Dt}\left(1 - \frac{u}{c}\right) + c\,h\left(1 + \frac{u}{c}\right) - r\,c\,\frac{Du}{Dt}\left(1 - \frac{u}{c}\right) - \frac{3}{2} c\,u^2\left(1 - \frac{u}{3c}\right) = 0 \;. \quad (17)$$

If, instead of Eq. (8), the quasi-acoustic approximation that $r\,(h + u^2/2)$ is propagated with velocity c_∞ were made and a similar mathematical analysis were carried out, as has been done in slightly different fashion by Trilling[5], the resulting differential equation would be

$$r \frac{Dh}{Dt}\left(1 - \frac{u}{c_\infty} + \frac{u^2}{c_\infty^2}\right) + c_\infty h - r\,c_\infty \frac{Du}{Dt}\left(1 - \frac{2u}{c_\infty}\right) - \frac{3}{2} c_\infty u^2\left(1 - \frac{4u}{3c_\infty}\right)$$

$$= 0 \;. \qquad (18)$$

On multiplying Eq. (17) by $(1/c)$ and Eq. (18) by $(1 + u/c_\infty)/c_\infty$, they are seen to agree up to first-order terms in u/c (since $c \approx c_\infty$ to zero order), but not to second-order terms*. Since Eq. (17) is presumably more accurate than Eq. (18), it is reasonable to suppose that Eq. (17) is accurate at least to terms in $(u/c)^2$, although this supposition has not been proved. The basic assumption, that $r\,(h + u^2/2)$ is propagated with a velocity $c + u$, is only partially justified by the short-wavelength argument (above) and by a related argument of Kirkwood and Bethe[7] for underwater explosions. The only presently known method of determining the accuracy of this hypothesis is to solve a number of flow problems more exactly, by numerical methods, and

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*In conventional acoustics, $u^2/2$ is neglected compared to h. If this approximation were made in the present situation, the result would contain erroneous first-order terms in u/c.

compare the answers with the above expressions. In a later section it will be shown that relations derived from Eq. (17) agree quite well with the only available numerical solution, which covers a velocity range extending up to $u/c = 2.2$. This agreement indicates that, at least in certain cases, the error introduced by the Kirkwood-Bethe hypothesis is not only less than $(u/c)^2$, but also small even when u/c is so large that the expansion of equations in powers of u/c becomes a dubious procedure.

III. THE BUBBLE-WALL MOTION

A. Equations Without the Subsonic Approximation

Since the growing or collapsing bubble wall is a "particle path", the particle-derivative relations derived in the previous section can be used to determine the variation of bubble radius with time. Let the capital letters R, U, C, H denote the values which the previously defined parameters (r, u, etc.) assume in the liquid at the bubble wall, and designate time derivatives by dots. Then Eq. (17) applied at the bubble wall becomes, when divided by C,

$$R\ddot{R}\left(1 - \frac{\dot{R}}{C}\right) + \frac{3}{2}\dot{R}^2\left(1 - \frac{\dot{R}}{3C}\right) = H\left(1 + \frac{\dot{R}}{C}\right) + \frac{R\dot{H}}{C}\left(1 - \frac{\dot{R}}{C}\right). \quad (19)$$

Equation (19) can also be written in terms of R and U, with the help of the relation $U\,dt = dR$:

$$RU\frac{dU}{dR}\left(1 - \frac{U}{C}\right) + \frac{3}{2}U^2\left(1 - \frac{U}{3C}\right) = H\left(1 + \frac{U}{C}\right) + \frac{RU}{C}\frac{dH}{dR}\left(1 - \frac{U}{C}\right). \quad (20)$$

In order to solve Eq. (19) or (20), it is necessary to know H and C as functions of t or R. In a typical physical situation, the bubble-wall pressure, P, is specified as a function of t or R. It is then necessary to use an equation of state relating H and C to P. For most liquids, it is found experimentally that the pressure-density curve for isentropic ("adiabatic") compression can be fitted closely by the formula

$$\frac{p + B}{p_\infty + B} = \left(\frac{\rho}{\rho_\infty}\right)^n , \tag{21}$$

where B and n are constants which depend upon the particular liquid under consideration (for water, $B \approx 3000$ atm and $n \approx 7$). From Eq. (21) it is readily found that

$$c^2 = \frac{dp}{d\rho} = \frac{n(p + B)}{\rho} = \frac{n(p + B)}{\rho_\infty}\left(\frac{p + B}{p_\infty + B}\right)^{-1/n} , \tag{22}$$

and hence

$$C = c_\infty\left(\frac{P + B}{p_\infty + B}\right)^{\frac{n-1}{2n}} \tag{23}$$

H can be evaluated with the help of Eqs. (5) and (21):

$$H = \int_{p_\infty} \left(\frac{p + B}{p_\infty + B}\right)^{-1/n} \frac{dp}{\rho_\infty} = \frac{n(p_\infty + B)}{(n - 1)\rho_\infty}\left[\left(\frac{P + B}{p_\infty + B}\right)^{\frac{n-1}{n}} - 1\right] . \tag{24}$$

If the liquid has an appreciable viscosity, μ, or surface tension, σ, the pressure P in the liquid at the bubble wall does not equal the pressure P_i exerted on the bubble wall by any interior gas or vapor, but the two pressures are related by the equation

$$P = P_i - \frac{2\sigma}{R} + 2\mu\left(\frac{\partial u}{\partial r}\right)_{r=R} - \frac{2\mu}{3}(\nabla \cdot \overline{u}) . \tag{25}$$

The partial derivative in Eq. (25) may be evaluated by Eq. (12), giving

$$P = P_i - \frac{2\sigma}{R} - \frac{4\mu U}{R} - \frac{4\mu U}{3C^2}\frac{dH}{dR} . \tag{26}$$

The last term on the right-hand side of Eq. (26) is small if either the viscosity or compressibility of the liquid is small, and it can be shown to be of the same order of magnitude as the viscous term neglected in Eq. (2). Omitting this term for consistency, one obtains

$$P = P_i - \frac{2\sigma}{R} - \frac{4\mu U}{R} . \tag{27}$$

When P_i is a specified function of t or R, Eq. (27) gives P as a function of U or R, and Eqs. (23) and (24) yield expressions for C and H which may then be substituted into the basic differential equation, (19) or (20). Thus, these relations can be combined into a single ordinary differential equation for the bubble wall motion, although the combined equation is rather lengthy and will not be reproduced here. This differential equation is nonlinear, and can be solved analytically only in special cases. For other situations, numerical methods must be used. The numerical solution of this ordinary differential equation, however, is very much simpler than the numerical solution of the standard partial differential equations for compressible flow.

Probably the simplest initial condition for a bubble growth or collapse problem is the condition of uniform pressure and zero velocity throughout the liquid up to $t = 0$, when the internal pressure in the bubble is suddenly changed to a new constant value. During the instant from $t = 0$ to $t = 0+$, the parameters P, H and C change discontinuously, and it will be shown that this produces a finite velocity jump in an infinitesimal time. Referring back to the general differential Eq. (20), one notes that the terms containing dU/dR and dH/dR are infinitely greater than the other terms, so that the equation can be simplified to

$$RU\,dU\,(1 - U/C) = (RU/C)\,dH\,(1 - U/C) . \tag{28}$$

Cancellation of common factors and integration yields

$$U_{o+} = \int_0^H \frac{dh}{c} \approx \frac{H}{C} \approx \frac{P - p_\infty}{\rho_\infty c_\infty} . \tag{29}$$

The exact part of Eq. (29) agrees with that obtained by Riemann's method for finite-amplitude plane waves[2]. The approximate expressions are accurate when $|H| << C^2$.

If, during the rest of the motion, the interior pressure P_i is constant (or zero) and viscous and surface-tension effects are small enough to be negligible, the bubble-wall equation can be solved analytically. In this situation Eqs. (27), (23) and (24) show that P, C and H are constant, and $dH/dR = 0$. The variables in Eq. (20) can then be separated to give, on integration:

$$\log_e \frac{R}{R_o} = -2 \int_{U_{o+}}^{U} \frac{U(U - C)dU}{U^3 - 3CU^2 + 2HU + 2HC}, \qquad (30)$$

where R_o is the initial radius and U_{o+} is given by Eq. (29). For given values of C and H, the integral in Eq. (30) may be evaluated numerically or graphically. Alternatively, a root of the cubic in the denominator may be obtained numerically, and the integrand separated into partial fractions which can be integrated analytically.

For the case of a bubble <u>collapsing</u> under a constant pressure difference, $P_i - p_\infty < 0$, $H < 0$ and $U < 0$. Under these conditions, neither the numerator nor the denominator of the integrand changes sign or vanishes in the range of integration. It follows that the magnitude of the bubble-wall velocity increases monotonically as the radius decreases, and becomes infinite at the point of collapse $(R \rightarrow 0)$, assuming that the Kirkwood-Bethe hypothesis still holds. More specifically, for small R and large $|U|$, the integral in Eq. (30) behaves like $-2\int(U^2dU/U^3)$, and hence $|U|$ behaves like $R^{-1/2}$.

In most bubble-collapse situations of interest, $|H| << C^2$ (for water, this inequality corresponds to $|P_i - p_\infty| << 20,000$ atm). In these situations, Eq. (30) can be approximated by

$$\log_e \frac{R}{R_o} = -2 \int_{U_{o+}}^{U} \frac{U[U - C - 4H/(27C)]dU}{U^3 - 3CU^2 - 2HU/3 + 2HC}, \qquad (31)$$

which can then be separated into partial fractions and integrated:

$$\log_e \frac{R}{R_o} = -\frac{2}{3}\int_{U_{o+}}^{U}\left(\frac{2}{U - 3C} + \frac{U - 4H/(9C)}{U^2 - 2H/c}\right)dU =$$

$$-\frac{1}{3}\left[4 \log (3C - U) + \log (U^2 + 2|H|/3) - \frac{4}{3}\sqrt{\frac{2|H|}{3C^2}} \tan^{-1}\left(U\sqrt{\frac{3}{2|H|}}\right)\right]_{U_{o+}}^{U}$$

(32)

where $|H|$ is written for $-H$, since H is negative. The relative error introduced by this approximation can be shown to less than $0.55 \sqrt{|H|/C^2}$ for all ranges of U/C, as long as U is negative. It may be noted that, to a similar approximation,

$$|H| = |P_i - p_\infty|/\rho_\infty \cdot \qquad (33)$$

Except during the very first part of collapse ($R_o - R << R_o$), the arctangent term in Eq. (32) can be neglected compared to the other terms, and U_{o+} can be neglected compared to U, so that Eqs. (32) and (33) can be combined to give the simple result

$$\left(\frac{R_o}{R}\right)^3 = \left(1 - \frac{U}{3C}\right)^4 \left(1 + \frac{3\rho_\infty U^2}{2(p_\infty - P_i)}\right) \cdot \qquad (34)$$

Neglect of the $U/3C$ term in Eq. (34) would yield the well known solution for a bubble collapsing in an incompressible liquid.[1, 2] It is seen from Eq. (34) that as $R \to 0$, the incompressible theory gives $U \sim R^{-3/2}$, while the present theory gives $U \sim R^{-1/2}$.

Figure 1 shows the theoretical variation of velocity with radius, for a bubble in water, collapsing under a constant pressure difference of $p_\infty - P_i = 0.517$ atm. The solid line gives values computed from

Eq. (34), while the dash-dot and the dashed curves represent the incompressible-flow solution and Herring's first-order compressible solution (see Section B below). As expected, the three curves approach each other at the low-velocity end, but diverge as U/C becomes large. Several points are also plotted which represent the results of Schneider's numerical integration of the partial differential equations for compressible flow.[8] Unfortunately, Schneider used as initial conditions for his numerical work the incompressible solution at $R/R_o = 0.08$, which gives a velocity significantly too high. The agreement of Schneider's calculations with the present theory is well within the limits of error of his calculations. It should be mentioned that all of these curves neglect viscosity and surface tension.

A set of velocity curves for bubbles collapsing in water, calculated from Eq. (34) with various values of the pressure difference, are presented in Fig. 2.

For purposes of comparison with the present theory, numerical calculations similar to Schneider's, but carried out with greater accuracy and extending to higher velocities, have been begun at the C.I.T. Hydrodynamics Laboratory. Such calculations are extremely laborious, and results will not be available for some time.

In the case of a bubble expanding under a constant positive pressure, with $H > 0$, $U > 0$, the analytic solution given by Eq. (32) or (34) is usually a poor approximation, because the denominator of the integrand in Eq. (30) has one or more zeros in the positive velocity range. When $0 < H < C^2/2$, the integrand has a non-integrable singularity at a certain value of U smaller than C. This value, which acts as a limiting value, can be expressed by the series.

$$U_{lim} = \left(\frac{2H}{3}\right)^{1/2} + \frac{2}{3C}\left(\frac{2H}{3}\right) + \frac{4}{9C^2}\left(\frac{2H}{3}\right)^{3/2} + \dots \qquad (35)$$

A bubble with a constant internal pressure in this range will, if suddenly released in a nonviscous liquid at rest, instantaneously attain the bubble-wall velocity given by Eq. (29), and then the bubble-wall

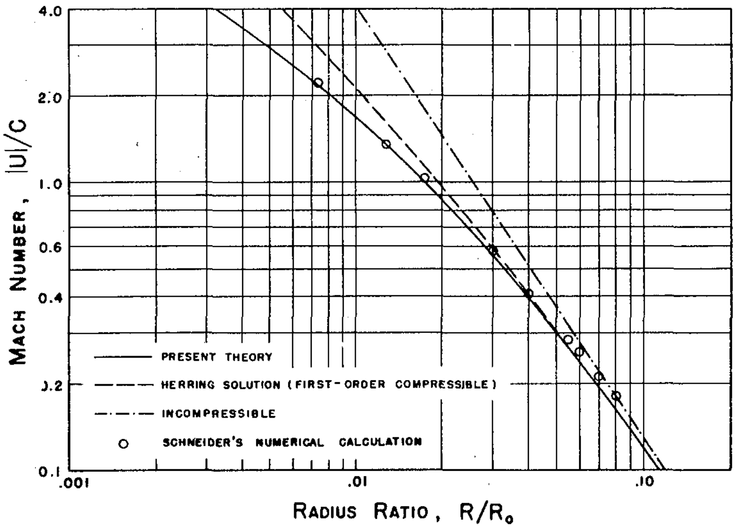

Fig. 1. The theoretical wall velocity of a bubble in water, collapsing under a constant pressure difference of 0.517 atmospheres.

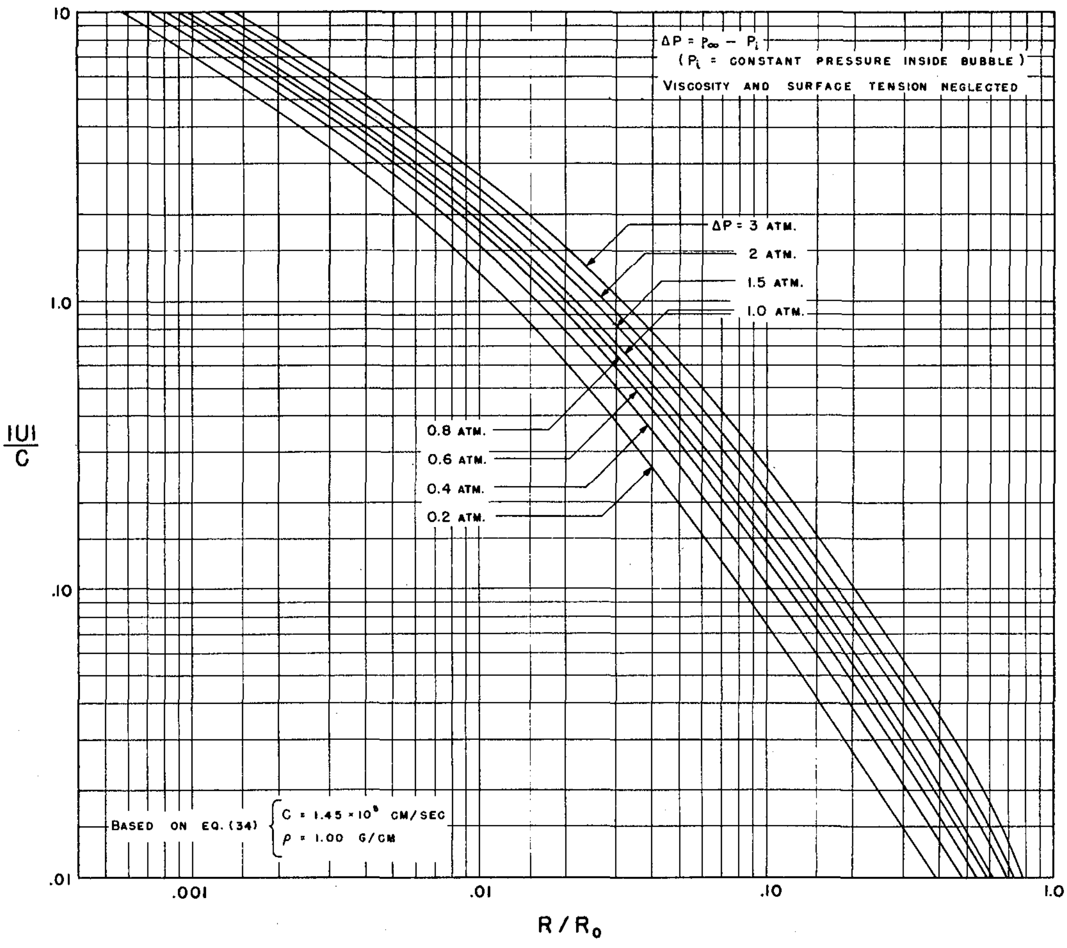

Fig. 2. The theoretical wall velocity of bubbles in water, collapsing under various constant pressure differences.

velocity will increase monotonically, approaching asymptotically the limiting value given by Eq. (35). Since this limiting velocity is less than C, the relations derived below for subsonic flow can be used to determine the detailed variation of velocity with radius. It may be noted that the first term on the right-hand side of Eq. (35) agrees with the limiting velocity derived for an expanding bubble in an incompressible liquid.

For a range of very high bubble pressures, $C^2/2 < H < C^2$, the integrand of Eq. (30) has no singularity in the subsonic range. The bubble-wall velocity will jump initially to the subsonic value given by Eq. (29), and then increase monotonically to $U = C$. At this point, Eq. (30) yields contradictory results, because it shows that R reaches a maximum there, whether the velocity thereafter increases, decreases, or remains constant. However, R obviously cannot have a maximum when $dR/dt = U = C$. If one puts $U = C$ in the basic differential relation, Eq. (20), the equation reduces to $H = C^2/2$, which disagrees with the assumed values of H. Evidently the present theory is not applicable near $U = +C$. This result is not completely unexpected, since it was found in Section II that the proof of the accuracy of the present theory for small wavelengths breaks down near $u = +c$.

For still higher bubble pressures, $H > C^2$, the bubble-wall velocity jumps initially to a supersonic value and then, according to Eq. (30), decreases monotonically to the value $U = C$, where the same mathematical difficulties then arise. In spite of these difficulties, it is believed that the decrease from very high velocities to velocities near sonic, as given by Eq. (30), is accurate. Only in the sonic region is the theory invalid, so that the actual behavior of a bubble expanding with near-sonic velocity is still not determined.

B. Approximate Equations for Subsonic Velocities

When the pressure inside the bubble is variable, or when viscosity and surface tension are taken into account, exact analytic solutions of the differential equation for the bubble-wall velocity cannot be found. However, if $|U| < C$, approximate solutions can be obtained by neglecting certain powers of the ratio U/C compared to unity. Specifically, when Eq. (20) is multiplied by $2 R^2 (1 - U/C + U^2/3 C^2) dR$ and terms proportional to $(U/C)^3$ and $(U/C)^4$ are neglected, the result can be integrated to give

$$R^3 U^2 \left(1 - \frac{4U}{3C} + \frac{2U^2}{C^2}\right) =$$

$$K_o + 2 \int_{R_o}^{R} \left[H \left(1 - \frac{2U^2}{3C^2}\right) + R \frac{dH}{dR} \left(\frac{U}{C} - \frac{2U^2}{C^2}\right)\right] R^2 \, dR, \tag{36}$$

where

$$K_o = R_o^{\;3} U_o^{\;2} \left(1 - \frac{4U_o}{3C_o} + \frac{2U_o^{\;2}}{C_o^{\;2}}\right). \tag{37}$$

In writing Eq. (36), certain terms containing dC/dR which are needed to make the left-hand side integrable are neglected on the right because they can be shown to be proportional to $(U/C)^3$, with the help of Eq. (22).

If P_i is a known function of R, C(R) and H(R) may be calculated with the help of Eqs. (23), (24) and (27), and then Eq. (36) can be used to find U(R). The procedure is not completely straightforward, since both sides of Eq. (36) contain both R and U. However, the U-dependent terms on the right are small, and can be neglected to get a first approximation for U(R), which then can be substituted into the small terms to yield a second approximation, etc., until an answer with the desired accuracy is obtained. Once U(R) is found, R(t) and U(t) can be calculated with the help of the relation $t = \int dR/U(R)$.

If P_i is known only as a function of time instead of radius, Eq. (36) is not useful, and recourse must be made to the basic differential Eq. (19) containing time-derivatives explicitly.

Equation (36) is valid only when $(U/C)^3$ is negligible. It is evident physically that such an upper bound placed on the velocity corresponds roughly to an upper bound on the magnitude of the pressure difference, $P - p_\infty$, provided that certain exceptional cases, such as pressures oscillating too rapidly to permit build-up of large velocities, are excluded. Consideration of the lowest-order (C-independent) terms in Eq. (36) reveals that H is usually of the same order as U^2. Consequently, from Eqs. (22) and (24), $(P - p_\infty)/(p_\infty + B)$ is of order U^2/C^2. It is then convenient to write $(P + B)/(p_\infty + B) = 1 + (P - p_\infty)/(p_\infty + B)$ in Eqs. (23) and (24), and expand them in power series:

$$C = c_\infty \left[1 + \frac{(n - 1)(P - p_\infty)}{2n\,(p_\infty + B)} + \ldots \right], \tag{38}$$

$$H = \frac{P - p_\infty}{\rho_\infty} \left[1 - \frac{P - p_\infty}{2n\,(p_\infty + B)} + \ldots \right], \tag{39}$$

where the neglected terms are usually of order $(U/C)^4$. Substitution of these values into Eq. (36) yields, to terms in $(U/C)^2$,

$$R^3\, U^2 \left(1 - \frac{4U}{3c_\infty} + \frac{2U^2}{c_\infty^2} \right) = K_o +$$

$$\frac{2}{\rho_\infty} \int_{R_o}^{R} \left[\left(P - p_\infty \right) \left(1 - \frac{2U^2}{c_\infty^2} - \frac{P - p_\infty}{2n\,(p_\infty + B)} \right) + R\,\frac{dP}{dR} \left(\frac{U}{c_\infty} - \frac{2U^2}{c_\infty^2} \right) \right] R^2\, dR \tag{40}$$

When the second-order terms, proportional to $(U/C)^2$ and $(P - p_\infty)/(p_\infty + B)$, are dropped from Eq. (40), and the effects of viscosity and surface tension are neglected so that $P = P_i$, the result agrees with that obtained by Herring[3, 4] and Trilling[5] using different methods, except that Herring inconsistently retained one second-order term, $-(RU^2/\rho_\infty c_\infty^2)(dP_i/dR)$, which is evidently in error by a factor of two. When both first and second-order terms in Eq. (40) are neglected, one obtains the well-known incompressible solution.[1, 2]

The effects of viscosity and surface tension may be exhibited explicitly by substituting Eq. (27) into Eq. (40) to get

$$R^3 U^2 \left(1 - \frac{4U}{3c_\infty} + \frac{2U^2}{c_\infty^2}\right) = K_o +$$

$$\frac{2}{\rho_\infty}\int_{R_o}^{R}\Bigg[\left(P_i - p_\infty\right)\left(1 - \frac{2U^2}{3c_\infty^2} - \frac{P_i - p_\infty}{2n\,(p_\infty + B)}\right) + R\,\frac{dp_i}{dR}\left(\frac{U}{c_\infty} - \frac{2U^2}{c_\infty^2}\right) -$$

$$\frac{2\sigma}{R}\left(1 - \frac{U}{c_\infty}\right) - \frac{2U}{c_\infty}\frac{d\sigma}{dR} - \frac{4\mu U}{R}\left(1 - \frac{U}{c_\infty}\right) - \frac{4U^2}{c_\infty R}\frac{d\mu}{dR} - \frac{4\mu U}{c_\infty R}\frac{dU}{dR}\Bigg] R^2\, dR \qquad (41)$$

In Eq. (41) terms proportional to μ/c_∞^2 have been dropped since similar terms were neglected in the derivation of the basic equations, while terms proportional to σ/c_∞^2 have been omitted for reasons of simplicity only. Equation (41) is not accurate, therefore, when the effects of both compressibility and viscosity or surface tension are large. When μ and σ are constant, Eq. (41) can, of course, be simplified by omission of their derivatives.

Because of the factor R^2 under the integral of Eq. (41), the various terms in the integrand are proportional to R, R^2 and R^3. Consequently, if the radius of a bubble varies by one or more orders of magnitude, the growth or collapse history of the bubble is determined principally by the pressure, viscosity and surface tension acting during the period when the bubble is at its largest size, provided that the order of magnitude of these three quantities does not change enough to overcome the radius effect. Moreover, the effect of initial conditions on the bubble-wall velocity is given by the term K_o/R^3, and thus becomes unimportant as a bubble grows to several times it original radius, while the initial conditions are important, of course, for collapsing bubbles.

C. Criteria for the Neglect of Viscosity and Surface Tension

Since the equations for a growing or collapsing bubble are considerably simplified by a neglect of the effects of viscosity and surface tension, it is desirable to determine when this neglect is permissible. Except for higher-order viscosity-compressibility interaction terms neglected throughout this report, the viscosity and surface tension act only to modify the effective pressure at the bubble wall, as given by Eq. (27). Consequently, these effects are negligible if

$$\left|\frac{2\sigma}{R} + \frac{4\mu U}{R}\right| << |P_i - p_\infty| \qquad (42)$$

throughout the motion. Moreover, as pointed out above, the bubble-wall motion is usually determined primarily by conditions existing during the period when the bubble is at its largest size. Hence, one can usually substitute the maximum value of R in the inequality (42), and find that the viscous and surface-tension effects are smaller than might otherwise be expected. To be on the safe side, however, one must substitute the maximum expected value of U in (42), instead of the value near R_{max}.

The effects of viscosity and surface tension on the collapse of a bubble under a constant pressure difference must be treated separately, because $U \rightarrow -\infty$ as $R \rightarrow 0$. To determine these effects, one may derive from Eqs. (24) and (27) the expansion

$$H = H_i - \frac{2\sigma + 4\mu U}{\rho_\infty R} + \ldots , \qquad (43)$$

where H_i is the enthalpy at $P = P_i$ (i.e., neglecting viscosity and surface tension), and higher order terms are omitted. Substitution of Eq. (43) in Eq. (20) yields, when H_i, μ and σ are constant,

$$RU\frac{dU}{dR}\left(1-\frac{U}{C}\right)+\frac{3}{2}U^2\left(1-\frac{U}{3C}\right)=H_i\left(1+\frac{U}{C}\right)$$

$$-\frac{2\sigma+4\mu U}{\rho_\infty R}\left(1+\frac{U^2}{C^2}\right)-\frac{4\mu U}{\rho_\infty C}\frac{dU}{dR}\left(1-\frac{U}{C}\right). \tag{44}$$

This equation can be rearranged and integrated formally to give an expression similar to Eq. (30):

$$\log_e\frac{R}{R_o}=2\int_{U_{o+}}^{U}\frac{(1+4\mu/\rho_\infty RC)(C-U)\,U\,dU}{U^3-3CU^2+2H_iU+2H_iC-\dfrac{4(2\mu U+\sigma)(C^2+U^2)}{\rho_\infty RC}} \tag{45}$$

Equation (45) is not very useful for determining the exact variation of velocity with radius, because R appears on both sides of the equation. However, it can be used to estimate the effects of viscosity and surface tension, as long as these effects are small.

It is convenient to restrict the discussion to situations where the pressure difference causing collapse is not extremely high (e.g., much less than 20,000 atm in water), so that $|H_i| << C^2$, and to divide the collapse period into low, medium and high-velocity intervals. The low-velocity interval will be defined as that during which $U^2 \leqq H_i$. During this interval the principal term in the denominator of Eq. (45) is $2H_iC$, and the relative error introduced in the integral of Eq. (45) by neglecting μ and σ is approximately

$$\epsilon\approx-\frac{4\mu U+2\sigma}{\rho_\infty R|H_i|}+\frac{4\mu}{\rho_\infty RC},\quad (U^2\leqq|H_i|)\,. \tag{46}$$

where the last term in Eq. (46) comes from the numerator of Eq. (45). According to Eq. (34), the bubble radius decreases from R_o to $(2/5)^{1/3}R_o$ during this low-velocity interval. Replacement of R in Eq. (46) by its minimum value and $|U|$ by its maximum gives an upper bound for the relative error:

$$|\epsilon| < \frac{6\mu}{\rho_\infty R_o |H_i|^{1/2}} + \frac{3\sigma}{\rho_\infty R_o |H_i|}, \quad (U^2 < |H_i|) \tag{47}$$

where the second viscous term has been dropped because it is negligible compared to the first.

The medium-velocity interval will be defined by $|H_i| \lessapprox U^2 \lessapprox C^2$. The denominator of Eq. (45) is larger in magnitude than $3CU^2$. Hence

$$|\epsilon| < \frac{4}{3\rho_\infty R}\left(\frac{2\mu}{|U|} + \frac{\sigma}{U^2} + \frac{2\mu|U|}{C^2} + \frac{\sigma}{C^2} + \frac{3\mu}{C}\right), \quad (|H_i| \lessapprox U^2 \lessapprox C^2) \ . \tag{48}$$

Replacement of U in the above terms by its maximum or minimum value (depending upon whether it appears in the numerator or denominator), R by its minimum value $(3/4)(H_i/2C^2)^{1/3} R_o$ from Eq. (34), and neglect of smaller terms yields

$$|\epsilon| < \frac{16}{9\rho_\infty R_o}\left(\frac{|H_i|}{2C^2}\right)^{1/3}\left(\frac{2\mu}{|H_i|^{1/2}} + \frac{\sigma}{|H_i|}\right), \quad (|H_i| \lessapprox U^2 \lessapprox C^2) \ . \tag{49}$$

Since $|H_i| << C^2$, it is evident that the relative error in the moderate-velocity interval is smaller in magnitude than that in the low-velocity interval.

In the high-velocity interval, defined by $U^2 \gtrapprox C^2$, the relative error introduced in the denominator of Eq. (45) includes a factor $(C^2 + U^2)/(|U|^3 + 3CU^2)$ which is less than $1/|U|$ throughout this interval, and hence the total relative error can be bounded by

$$|\epsilon| < \frac{12\mu}{\rho_\infty RC} + \frac{4\sigma}{\rho_\infty RC^2}, \quad (U^2 \geqq C^2) \,. \tag{50}$$

Since $R \to 0$ in this interval, this upper bound becomes infinite near the point of collapse. In fact, it can be shown that the influence of viscosity necessarily becomes large very near the point of collapse. In many practical situations, however, this region of viscous influence corresponds to bubble radii too small to be observable.

In view of Eqs. (47), (49) and (50), the relative error in the value of $\log_e R/R_o$ calculated for a given value of U can be bounded by

$$|\epsilon| < \frac{6\mu}{\rho_\infty}\left(\frac{1}{R_o\,|H_i|^{1/2}} + \frac{2}{RC}\right) + \frac{\sigma}{\rho_\infty}\left(\frac{3}{R_o|H_i|} + \frac{4}{RC^2}\right) \tag{51}$$

for all velocity ranges. This upper bound can also be written in terms of the pressure difference, with the help of Eq. (33):

$$|\epsilon| < 6\mu\left(\frac{1}{R_o\sqrt{(p_\infty - P_i)\rho_\infty}} + \frac{2}{\rho_\infty RC}\right) + \sigma\left(\frac{3}{R_o(p_\infty - P_i)} + \frac{4}{\rho_\infty RC^2}\right) . \tag{52}$$

Application of Eq. (52) to bubbles in water at ordinary temperatures shows that the error introduced by neglecting viscosity and surface tension is less than 1% for bubbles having $R_o \geqq 1$ mm and $p_\infty - P_i \geqq 0.3$ atm, up to the point where they have collapsed to $R = 2 \times 10^{-3}$ mm. The 1% figure refers to the relative error in $\log_e R/R_o$; this figure corresponds to a relative error in R of less than 1% when $R/R_o > 0.37$ and greater than 1% when $R/R_o < 0.37$. During the first part of the collapse, the surface-tension effects are much larger than the viscous effects, while during the later part of collapse viscous effects predominate.

The value given by Eq. (52) represents an upper bound on the effect of viscosity and surface tension; in many practical situations it is 2 to 10 times the actual effect. In border-line situations, it may be desirable to get a more exact estimate. For any specific numerical initial conditions, this estimate may be made quite readily with the help of Eq. (45). No general treatment will be included here, as such a treatment would be greatly complicated by the necessity of considering separately many different cases, depending upon the relative magnitude of the parameters involved.

Throughout this section, the variation of sonic velocity caused by viscosity and surface tension changing the pressure at the bubble wall has been neglected. This effect is always much smaller than the effects considered above, since the difference between C and C_i is of higher order than that between H and H_i, and, moreover, in the low-velocity interval the major terms in both numerator and denominator of Eq. (45) are proportional to C, so that any variation cancels out, while in the high-velocity interval the major terms are independent of C.

IV. VELOCITY AND PRESSURE FIELDS THROUGHOUT THE LIQUID

A. The Quasi-Acoustic Approximation

In deriving relations for the velocity and pressure fields throughout the liquid, it is convenient to work first with the simple quasi-acoustic equations, as this work then serves to guide the investigation of higher-order approximations. In addition, the quasi-acoustic approximation, which is accurate to terms of the first order in u/c, is adequate for many practical purposes.

In the quasi-acoustic approximation, the radial velocity is found by combining Eqs. (1) and (6):

$$u = \frac{f(t - r/c_\infty)}{r^2} + \frac{f'(t - r/c_\infty)}{r\, c_\infty} \quad . \tag{53}$$

With the help of the first-order approximation $h = (p - p_\infty)/\rho_\infty$, Eq. (7) can be written

$$p = p_\infty + \frac{\rho_\infty f'(t - r/c_\infty)}{r} - \frac{\rho_\infty u^2}{2} \; . \tag{54}$$

When Eqs. (53) and (54) are solved simultaneously for f and f', the results are

$$f(t - r/c_\infty) = \frac{r^2}{c_\infty}\left(c_\infty u - \frac{u^2}{2} - \frac{p - p_\infty}{\rho_\infty}\right) \; ; \tag{55}$$

$$f'(t - r/c_\infty) = r\left(\frac{u^2}{2} + \frac{p - p_\infty}{\rho_\infty}\right) \; . \tag{56}$$

Evaluation of Eqs. (55) and (56) at the bubble wall, where $r = R$, $u = U$ and $p = P = P_i - 2\sigma/R - 4\mu U/R$ from Eq. (27), yields

$$f(t - R/c_\infty) = \frac{R^2}{c_\infty}\left(c_\infty U - \frac{U^2}{2} - \frac{P_i - p_\infty}{\rho_\infty} + \frac{2\sigma}{\rho_\infty R} - \frac{4\mu U}{\rho_\infty R}\right) \; ; \tag{57}$$

$$f'(t - R/c_\infty) = \frac{RU^2}{2} + \frac{R(P_i - p_\infty)}{\rho_\infty} - \frac{2\sigma}{\rho_\infty} - \frac{4\mu U}{\rho_\infty} \; . \tag{58}$$

If P_i, σ and μ are either constants or known functions of R or t, U(R) and t(R) may be found by the methods of Section III. By evaluating Eqs. (57) and (58) for various values of R, numerical values of f and f' can be found for any desired range of the argument $t - R/c_\infty$. (Eq. (58) is actually superfluous, since values of f' could be obtained by numerical differentiation of f, but use of Eq. (58) is generally more convenient.) Once f and f' are determined numerically for a wide enough range of the argument, values of u and p throughout the liquid may be obtained from Eqs. (53) and (54). The entire process may be readily carried out numerically or graphically, even though the results can not be expressed by explicit equations.

If an explicit analytic solution is required, a further approximation must be made. Specifically, it may be assumed that the function f and its derivatives change little with a change in their argument by $(r - R)/c_\infty$, so that the first few terms in the Taylor expansion,

$$f(t - r/c_\infty) = f(t - R/c_\infty) - \left(\frac{r - R}{c_\infty}\right) f'(t - R/c_\infty) + \frac{1}{2}\left(\frac{r - R}{c_\infty}\right)^2 f''(t - R/c_\infty) - \ldots , \qquad (59)$$

provide an adequate approximation. With the use of this expansion, Eqs. (53) and (54) become

$$u = \frac{f}{r^2} + \frac{Rf'}{c_\infty r^2} - \frac{r^2 - R^2}{r^2} \frac{f''}{2c_\infty^2} + \ldots ; \qquad (60)$$

$$p = p_\infty + \frac{\rho_\infty f'}{r} - \frac{\rho_\infty f^2}{2r^4} - \frac{\rho_\infty (r - R) f''}{c_\infty r} - \frac{\rho_\infty R f f'}{c_\infty r^4} + \ldots \qquad (61)$$

In Eqs. (60) and (61) the argument $(t - R/c_\infty)$ has been omitted from f and its derivatives, for simplicity.

The second derivative f'' may be found with the help of Eq. (58) for f'. It is convenient to rewrite this equation in terms of P instead of P_i:

$$f'(t - R/c_\infty) = \frac{RU^2}{2} + \frac{R(P - p_\infty)}{\rho_\infty} . \qquad (62)$$

The derivative of Eq. (62) with respect to t is

$$(1 - U/c_\infty) f''(t - R/c_\infty) = \frac{U^3}{2} + RU\ddot{R} + \frac{U(P - p_\infty)}{\rho_\infty} + \frac{RU}{\rho_\infty}\frac{dP}{dR} , \qquad (63)$$

since $dR/dt = U$ and $dP/dt = U\,dP/dR$. $\ddot{R}$ may be evaluated from Eq. (19). The quasi-acoustic approximation is only accurate to the lowest order compressible terms, so that the first approximations to Eqs. (19) and (63) are adequate. When these are combined, one obtains

$$f''(t - R/c_\infty) = -U^3 + \frac{2U(P - p_\infty)}{\rho_\infty} + \frac{RU}{\rho_\infty}\frac{dP}{dR}. \qquad (64)$$

Substitution of the values for f, f' and f'' obtained above into Eqs. (60) and (61) yields, to the same approximation,

$$u = U\left[\frac{R^2}{r^2} + \left(\frac{r^2 - R^2}{r^2}\right)\left(\frac{U^2}{2c_\infty^2} - \frac{P - p_\infty}{\rho_\infty c_\infty^2} - \frac{R}{2\rho_\infty c_\infty^2}\frac{dP}{dR}\right)\right]; \qquad (65)$$

$$p = p_\infty + \frac{R}{r}\left(P - p_\infty\right) + \frac{R(r^3 - R^3)}{r^4}\left(\frac{\rho_\infty U^2}{2}\right)$$

$$+ \left(\frac{r - R}{r}\right)\left(\frac{U}{c_\infty}\right)\left[\rho_\infty U^2 - 2(P - p_\infty) - R\frac{dP}{dR}\right]. \qquad (66)$$

The first term in Eq. (65) and the first three terms in Eq. (66) represent the well-known velocity and pressure fields in an incompressible liquid.[1,2] The remaining terms give the lowest order correction for compressibility; they have apparently not been previously derived. Because these equations include only the first few terms in a Taylor expansion about the point $r = R$, they are accurate only in a region near the bubble wall. A consideration of the next terms in the expansions, not included here, shows that the relative error in these equations for u and p is usually of the order of $(U/c_\infty)^2 (r/R)^2$, so that when $U/c_\infty = 0.1$, for example, 10% accuracy is maintained out to a distance of about three times the bubble radius. At greater distances from the bubble wall, the relative error in u and $(p - p_\infty)$ will be greater; however, the magnitude of u and $(p - p_\infty)$ may be so small at these distances that the absolute error may still be negligible.

The effects of viscosity and surface tension on the velocity and pressure fields may be exhibited explicitly by substituting Eq. (27) in Eqs. (65) and (66). The resulting expressions are rather lengthy and will not be reproduced here.

In the above discussion, two methods have been presented for determining the liquid velocity and pressure fields in terms of the velocity and pressure at the bubble wall, when all velocities are considerably less than the sonic velocity. In some situations of interest, however, velocities near the bubble wall approach or exceed sonic velocity, so that these methods are not applicable. Even in these situations, velocities at points sufficiently far from the bubble will be small enough that the quasi-acoustic approximation is valid. In calculations of the flow around the bubble by the "method of characteristics" or similar numerical integration methods, considerable computational labor may be avoided by using the quasi-acoustic relations for regions in which velocities are small. For this purpose, two theorems which follow immediately from Eqs. (55) and (56) will be stated:

(a) In the quasi-acoustic approximation, the quantity $\left[c_\infty u - u^2/2 - (p - p_\infty)/\rho_\infty\right]$ is propagated outward with a velocity c_∞ and an amplitude decreasing with the inverse square of the radius.

(b) In the same approximation, the quantity $\left[u^2/2 + (p - p_\infty)/\rho_\infty\right]$ is propagated outward with a velocity c_∞ and an amplitude decreasing inversely with the radius.

If u and p are calculated, by the method of characteristics, for a point far enough from the bubble wall to be out of the high velocity region, the two quantities mentioned in (a) and (b) can be readily calculated for this point, and then extended to all other points farther from the bubble by means of the above theorems. Having evaluated these two quantities throughout the region of interest, one may readily solve for u and p throughout this region.

B. The Second-Order Approximation

All of the methods described above for obtaining the liquid velocity and pressure fields are based on the quasi-acoustic approximation, and are accurate only to first-order terms in u/c_∞. They suggest, however, a method by which second-order approximations may be obtained, provided that the Kirkwood-Bethe hypothesis is sufficiently accurate. This hypothesis states that a quantity which will be denoted here by y:

$$y(r, t) = r(h + u^2/2) , \tag{67}$$

is constant along any path traced by a point moving outward with the variable velocity $c + u$. Such a path is known as an "outgoing characteristic". If differentiation along an outgoing characteristic is denoted by d/dr, where

$$\frac{d}{dr} = \frac{\partial}{\partial r} + \frac{1}{c + u} \frac{\partial}{\partial t} , \tag{68}$$

the Kirkwood-Bethe hypothesis gives

$$\frac{dy}{dr} = 0 . \tag{69}$$

In the quasi-acoustic approximation the quantity

$$z(r, t) = c_\infty r^2 u - ry \tag{70}$$

also remains constant along an outgoing characteristic (see Theorem (a)). For a second-order approximation, the variation of z along the characteristic should be determined approximately. Differentiation of Eq. (70) gives

$$\frac{dz}{dr} = 2c_\infty ru + c_\infty r^2 \frac{du}{dr} - y , \tag{71}$$

since $dy/dr = 0$. The derivative du/dr may be evaluated from the basic continuity and momentum equations. When Eq. (12) is multiplied by c and added to Eq. (11), the result can be written

$$(c + u)\frac{du}{dr} = -\frac{2cu}{r} - \frac{c + u}{c}\frac{dh}{dr}, \tag{72}$$

which is equivalent to a well-known formula for integration along a characteristic. Upon solving Eq. (67) for h, introducing this value into Eq. (72), and solving the resulting equation for du/dr, one obtains

$$\frac{du}{dr} = -\frac{2c^2u}{(c^2 - u^2)r} + \frac{y}{(c - u)r^2}. \tag{73}$$

Substitution of this result for du/dr into Eq. (71) yields

$$\frac{dz}{dr} = -\frac{2c_\infty ru^3}{c^2 - u^2} + \frac{(c_\infty - c + u)y}{c - u}. \tag{74}$$

As expected, the relative rate of variation of z with r is small, being of the order of u^2/c^2 from Eq. (74). It is permissible, therefore, to use the lowest-order approximations to the terms on the right-hand side of Eq. (74). Since $(c_\infty - c)/c \sim u^2/c^2$ from Section III, this approximation is

$$\frac{dz}{dr} = -\frac{2ru^3}{c_\infty} + \frac{uy}{c_\infty}. \tag{75}$$

From Eq. (71) $u \approx z/c_\infty r^2$. With this substitution, Eq. (75) becomes

$$\frac{dz}{dr} = -\frac{2z^3}{c_\infty^4 r^5} + \frac{yz}{c_\infty^2 r^2}. \tag{76}$$

Although Eq. (76) is nonlinear, it can be made linear by taking $1/z^2$ as a new dependent variable. The solution is then found by standard methods to be

$$z = \frac{y^2}{c_\infty^2}\left[\frac{3}{2} + \frac{3y}{c_\infty^2 r} + \frac{3y^2}{c_\infty^4 r^2} + \frac{2y^3}{c_\infty^6 r^3} + K_1 \exp\left(\frac{2y}{c_\infty^2 r}\right)\right]^{-1/2}, \quad (77)$$

where K_1 is an arbitrary constant.

The exponential in Eq. (77) can be expanded in a power series to give

$$z = \frac{y^2}{c_\infty^2}\left[K_2\left(1 + \frac{2y}{c_\infty^2 r} + \frac{2y^2}{c_\infty^4 r^2} + \frac{4y^3}{c_\infty^6 r^3}\right) + \left(K_2 - \frac{3}{2}\right)\left(\frac{2y^4}{3c_\infty^8 r^4} + \ldots\right)\right]^{-1/2}, \quad (78)$$

where $K_2 = K_1 + 3/2$. It is seen from Eqs. (67) and (70) that the order of magnitude of $y/c_\infty^2 r$ is u^2/c_∞^2 and of $c_\infty^2 z/y^2$ is c_∞^3/u^3. It follows from Eq. (78) that K_2 is of the order u^6/c^6, and thus the approximation

$$z = \frac{y^2}{c_\infty^2}\left[K_2\left(1 + \frac{2y}{c_\infty^2 r}\right) - \frac{y^4}{c_\infty^8 r^4}\right]^{-1/2} = \frac{K_3 y^2}{c_\infty^2}\left(1 - \frac{y}{c_\infty^2 r} + \frac{K_3^2 y^4}{2c_\infty^8 r^4}\right), \quad (79)$$

where $K_3 = K_2^{-1/2}$, has a negligible relative error (of order u^4/c^4). The velocity may then be obtained from Eqs. (70) and (79)

$$u = \frac{y}{c_\infty r} + \frac{K_3 y^2}{c_\infty^3 r^2}\left(1 - \frac{y}{c_\infty^2 r} + \frac{K_3^2 y^4}{2c_\infty^8 r^4}\right) \quad (80)$$

The pressure may be expressed in terms of the enthalpy difference by inverting the series of Eq. (39). When the value of h from Eq. (67)

is substituted in the inverted series, one obtains

$$p - p_\infty = \rho_\infty\left(\frac{y}{r} - \frac{u^2}{2}\right) + \frac{\rho_\infty}{2c_\infty^2}\left(\frac{y}{r} - \frac{u^2}{2}\right)^2 . \tag{81}$$

Along any particular outgoing characteristic, the quantities y and K_3 are constants which may be determined if the velocity and pressure at one point along the characteristic are known. For example, if $U(R)$ and $P(R)$ at the bubble wall are determined, for a given instant, by the methods of Section III, the value of y along the corresponding outgoing characteristic is found, with the help of Eqs. (39) and (67), to be

$$y \doteq \frac{RU^2}{2} + \frac{R(P - p_\infty)}{\rho_\infty}\left(1 - \frac{P - p_\infty}{2\rho_\infty c_\infty^2}\right) . \tag{82}$$

The value of K_3 is determined by putting $u = U$ and $r = R$ in Eq. (80). A first approximation, $K_3 = c_\infty R^2 U/y^2$, is found by neglecting all but the lowest order terms. This value for K_3 may then be inserted into the small term, $K_3{}^2 y^4/2c_\infty^8 R^4$, and the resulting equation solved to obtain a value of K_3 accurate to second order:

$$K_3 = \frac{c_\infty^3 R^2 U}{y^2}\left(1 - \frac{U^2}{2c_\infty^2}\right) - \frac{c_\infty^2 R}{y}\left(1 - \frac{U}{c_\infty}\right). \tag{83}$$

Equations (80) and (81) with constants evaluated from Eqs. (82) and (83), yield values of the velocity and pressure along any outgoing characteristic, as a function of r. The time co-ordinate corresponding to these values can be determined from the equation

$$t = t_R + \int_R^r \frac{dr}{c + u} , \tag{84}$$

where t_R is the time at which the characteristic started at the bubble wall, $r = R$. In the quasi-acoustic approximation, the integral in Eq. (84) would be approximated by $\int dr/c_\infty$. In the second-order approximation, it is sufficient to use $\int (1 - u/c_\infty) dr/c_\infty$ and approximate u by the incompressible value, UR^2/r^2, so that Eq. (84) yields

$$t = t_R + \left(\frac{r - R}{c_\infty}\right)\left(1 - \frac{UR}{c_\infty r}\right) . \tag{85}$$

This completes the set of equations necessary for determining the velocity and pressure fields in the liquid to second-order accuracy.

The effects of viscosity and surface-tension enter only in the boundary condition, $P = P_i - 2\sigma/R - 4\mu U/R$, so that Eq. (82) can be written

$$y = \frac{RU^2}{2} + \frac{R(P_i - p_\infty)}{\rho_\infty}\left(1 - \frac{P_i - p_\infty}{2\rho_\infty c_\infty^2}\right) - \frac{2\sigma}{\rho_\infty} - \frac{4\mu U}{\rho_\infty} . \tag{86}$$

The direct effect of viscosity on the propagation of the pressure and velocity through the liquid is of order $\mu(u/c_\infty)^2$. This effect has been neglected throughout this paper because of its small magnitude (in the usual situation) and because of the great difficulties in its analytic treatment.*

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*It is remarkable that in the spherical flow of an incompressible liquid ($c_\infty \to \infty$), viscosity not only has no effect on the shape of the velocity field, which must, of course, vary with $1/r^2$, but it also has no effect on the pressure field corresponding to a given velocity field, even though energy is viscously dissipated throughout the liquid. (This energy is balanced by the additional work done at the boundary because of the difference between P and P_i.)

It may be noted that Eqs. (80), (81), (83), (85) and (86) permit straightforward numerical calculation of the velocities and pressures at a network of points covering (r, t) space as densely as desired, provided that the exact co-ordinates of the points are not required to be specified in advance. If, however, velocities and pressures are required at points specified initially, a process of successive approximation must be used to determine the bubble-wall values (R, t_R) at which the characteristics passing through the given points originate.

In certain circumstances, some of the outgoing characteristics may "catch up with" or cross characteristics which originated earlier at the bubble wall, because of their varying propagation velocities. At points beyond the crossover, the equations derived above will yield two different values for both the velocity and the pressure. In reality, the crossover of two characteristics signals the appearance of a shock wave, and the above relations then become invalid. However, if the energy dissipation in the shock wave has a negligible effect on the liquid density (which is the case for shocks involving pressure jumps up to 50, 000 atmospheres in water), the actual velocities and pressures will follow one branch of the calculated curves up to a certain point, then jump across to the other branch by means of the shock. In most spherical flow problems of present interest, the region where the equations give double values is small, since this region stops growing as soon as it has moved far enough away from the bubble for the velocities and pressures to diminish to values small enough that the quasi-acoustic approximation is valid. It is usually sufficiently accurate to locate the shock wave in the middle of the double-valued region. The derivation of a more accurate locating procedure will not be attempted in this report.

C. Equations Without the Subsonic Approximation

Since it appears that in certain situations the Kirkwood-Bethe hypothesis is accurate even when liquid velocities approach or exceed sonic velocity, it may be desirable to solve the equations for velocity and pressure without neglecting any powers of u/c. In this situation, the "exact" Eq. (73) or (74) must be solved, the former being the more convenient. For this purpose c is first expressed in terms of h by means of Eqs. (22), (23) and (24):

$$c = c_\infty \sqrt{\frac{(n-1)\rho_\infty h}{n(p_\infty + B)} + 1} = \sqrt{c_\infty^2 + (n-1)h}\ , \qquad (87)$$

and then in terms of u and y with the help of Eq. (67). When the result is substituted in Eq. (73), one obtains the differential equation for integrating along a characteristic:

$$\frac{du}{dr} = -\frac{2u}{r} - \frac{2u^3}{c_\infty^2 r - (n+1)ru^2/2 + (n-1)y}$$

$$+ \frac{y}{r\left(\sqrt{c_\infty^2 r^2 - (n-1)r^2u^2/2 + (n-1)ry} - ur\right)} . \qquad (88)$$

This equation is subject to the initial condition $u = U$ at $r = R$. Along any particular characteristic, y has the constant value

$$y = R(H + U^2/2) . \qquad (89)$$

No analytic solution of Eq. (88) is known, so it must be solved numerically for each initial condition and value of y. Once $u(r)$ is found, $h(r)$ can be evaluated from Eq. (67), and then $p(r)$ from Eq. (24). The corresponding values of t are found with the help of Eq. (84), which can be written

$$t = t_R + \int_R^r \frac{dr}{\sqrt{c_\infty^2 + (n-1)h} + u} . \qquad (90)$$

This entire procedure, though lengthy, may still be shorter than solving the basic partial differential equations for the flow numerically.

V. THE ACOUSTIC RADIATION

In a nonviscous liquid, the total rate at which energy crosses a spherical surface of radius r fixed in the liquid (where r varies with time, since $dr/dt = u$) is simply the rate at which work is done by the pressure:

$$\frac{dW_{tot}}{dt} = 4\pi r^2 pu \ , \tag{91}$$

since no mass crosses the surface thus defined. In a viscous liquid, an additional term should appear in Eq. (91) to account for the energy transmitted by viscous stresses. It has been shown in Section IV, however, that viscosity does not directly affect the velocity and pressure fields in the radial flow of an incompressible liquid, but only affects conditions at the boundary. It follows (or can be shown directly) that the energy transmitted in the incompressible liquid by viscous forces is exactly balanced by the energy viscously dissipated. Hence, this portion of the energy can be dropped from consideration, and Eq. (91) written for the remaining energy. In this sense, Eq. (91) is valid when either the viscosity or compressibility of the liquid is negligible, and it should be a reasonable approximation when both viscosity and compressibility are moderately small.

It is convenient to separate the energy flow given by Eq. (91) into two parts by writing

$$\frac{dW_{tot}}{dt} = p_\infty \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) + 4\pi r^2 (p - p_\infty) u \ . \tag{92}$$

If the fluid motion were periodic, as in the usual acoustic situation, and the energy flow were integrated over an integral number of periods, the contribution of the first term on the right-hand side of Eq. (92) would vanish, since it is proportional to the net displacement of the fluid. The remaining term may be called the "wave-energy" flow, so that

$$\frac{dW_{wave}}{dt} \equiv 4\pi r^2 (p - p_\infty) u \ . \tag{93}$$

In the quasi-acoustic approximation, substitution of values of u and p from Eqs. (53) and (54) into Eq. (93) yields

$$\frac{dW_{wave}}{dt} = 4\pi\rho_{\infty}\left[\frac{(f')^2}{c_{\infty}} + \frac{ff'}{r} + \frac{1}{2r}\left(\frac{f}{r} + \frac{f'}{c_{\infty}}\right)^3\right]. \qquad (94)$$

In Eq. (94) it is understood that f and f' are functions of $(t - r/c_{\infty})$. A long distance from the bubble, as $r \to \infty$, all but one of the terms in Eq. (94) become negligible. The remaining term gives what is usually defined as the acoustic radiation:

$$\frac{dW_{acoustic}}{dt} = \frac{4\pi\rho_{\infty}(f')^2}{c_{\infty}}. \qquad (95)$$

It should be emphasized that the three quantities denoted above by "total energy", "wave energy" and "acoustic energy" are, in general, distinct, and become equal only when the conditions of periodic motion, small amplitude, and large distance from the source are satisfied.

If the velocity and pressure at the bubble wall are known, the acoustic radiation which will appear, after a suitable lapse of time, at points far from the bubble can be calculated from Eqs. (58) and (95):

$$\frac{dW_{acoustic}}{dt} = \frac{4\pi R^2}{\rho_{\infty}c_{\infty}}\left[(P_i - p_{\infty}) + \frac{\rho_{\infty}U^2}{2} - \frac{2\sigma}{R} - \frac{4\mu U}{R}\right]^2. \qquad (96)$$

Equation (96) is based on the quasi-acoustic approximation.

In the second-order approximation, values of u and p from Eqs. (80) and (81) may be substituted in Eq. (93) to get a very lengthy expression for the wave energy flow, which will not be reproduced here. At long distances from the bubble, this expression reduces to the acoustic radiation:

$$\frac{dW_{acoustic}}{dt} = \frac{4\pi R^2}{\rho_{\infty}c_{\infty}}\left[(P - p_{\infty})\left(1 - \frac{P - p_{\infty}}{2\rho_{\infty}c_{\infty}^2}\right) + \frac{\rho_{\infty}U^2}{2}\right]^2, \qquad (97)$$

where Eq. (82) has been used to eliminate y. If desired, P may be expressed in terms of P_i, μ and σ with the help of Eq. (27).

If it is desired to find the total acoustic energy radiated during a certain part of the bubble motion, it must be realized that the duration, Δt, of the acoustic pulse at $r \rightarrow \infty$ is not quite equal to its duration, Δt_R, when it was at the bubble wall, since its velocity of propagation has changed. Differentiation of Eq. (85) with respect t_R and use of the relations $dr/dt = u$, $dR/dt_R = U$, yields

$$\frac{dt}{dt_R} = 1 + \frac{1}{c_\infty}\left(u\frac{dt}{dt_R} - U\right)\left(1 - \frac{UR}{c_\infty r}\right)$$

$$- \left(\frac{r - R}{c_\infty}\right)\left(\frac{\dot{U}R}{c_\infty r} + \frac{U^2}{c_\infty r} - \frac{URu}{c_\infty r^2}\frac{dt}{dt_R}\right). \qquad (98)$$

A long distance from the bubble $r \rightarrow \infty$ and $u \rightarrow 0$, so that Eq. (98) becomes simply

$$\frac{dt}{dt_R} = 1 - \frac{U}{c_\infty} - \frac{U^2}{c_\infty^2} - \frac{\dot{U}R}{c_\infty^2}. \qquad (99)$$

Equation (97) can then be integrated in the form

$W_{acoustic} =$

$$\frac{4\pi}{\rho_\infty c_\infty}\int R^2\left[(P - p_\infty)\left(1 - \frac{P - p_\infty}{2\rho_\infty c_\infty^2}\right) + \frac{\rho_\infty U^2}{2}\right]^2\left[1 - \frac{U}{c_\infty} - \frac{U^2}{c_\infty^2} - \frac{\dot{U}R}{c_\infty^2}\right]dt_R \qquad (100)$$

where t_R is the time co-ordinate in which R and U are usually expressed.

In situations involving near-sonic or supersonic velocities, neither the first nor second-order approximations are adequate, and the differential equations of Section IV C must be solved numerically to find the velocity and pressure fields from which the acoustic radiation may then be determined.

If any attempt is made to use the expressions for acoustic radiation together with the principle of conservation of energy to determine certain aspects of the bubble motion, it must be realized that only part of the flow energy is radiated outward with sonic velocity as acoustic energy; the rest is stored, for a longer or shorter period, in the neighborhood of the bubble. This phenomenon limits the usefulness of energy considerations. Although certain inequalities may still be derived by such methods, it appears simpler in most cases to use the basic equations for bubble-wall motion.

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